SDMC: Generating Functions-Problems

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Note: Some problems borrowed from Sean Markan. Also note that the problems are not necessarily in order of difficulty.

1. Find generating functions for the following sequences
   a) 1,0,1,0,1,0,1,...
   b) 1,2,3,4,5,6,...
   c) 1,3,6,10,15,21,...
   d) 1,4,9,16,25,...
   e) 1,8,27,64,125...

2. Find a generating function for the sequence defined as such:

$$a_0 = 0$$

$$a_{n+1} = 2a_n + 1$$

3. Using the generating founction found in the previous problem, find an explicit formula for a_n.

4. Find a generating function for the sequence defined by:

$$a_0 = 1$$

$$a_{n+1} = 2a_n + n$$

5. Using the generating function found in the previous problem, find an explicit formula for a_n.

6. Find a generating function for the number of different ways to make n cents out of pennies, nickels, dimes, quarters and half dollars.

7. A teacher wants to distribute 10 pieces of candy to 2 students. One student wil only take an even number of pieces of candy. The third student will take any number of pieces of candy. How many ways can the teacher distribute the candy? Can you find a generating function for n candies?

8. Show that the number of partitions of an integer n into parts the largest of which is r is equal to the number of partitions of n into exactly r parts.

9. Find generating functions for:
   a) The number of partitions of n where each part is different
   b) The number of partitions of n into odd parts
   c) The number of partitions of n with at most k parts
   d) The number of partitions of n with exactly k parts
   e) (AoPS Volume 2) the number of integer solutions of $2x+3y+7z = n$ with $z < 4$

10. Prove that the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.

11. (USAMO) Consider a partition p of n. Let $f(p)$ be the number of 1's in p and $g(p)$ be the number of distinct numbers in p. Prove that for all n, the sum of $f(p)$ over all the partitions p of n equals the sum of $g(p)$ over all the partitions p of n.

12. (Romania 1994) How many polynomials P with coefficients 0,1,2, or 3 satisfy $P(2) = n$, where n is a given positive integer?

13. (HMMT 2008) Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each k = 1,2,3,4, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_4 + 5b_4 = 19$.

14. (Putnam 1957) Show that the number of ways of representing n as an ordered sum of 1s and 2s equals the number of ways of representing $n + 2$ as an ordered sum of integers all greater than 1. For example:

$$4 = 1 + 1 + 1 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 2 + 1 = 1 + 1 + 2 (5 Ways)$$

$$6 = 4 + 2 = 2 + 4 = 3 + 3 = 2 + 2 + 2 (5 Ways)$$

15. (Art and Craft of Problem Solving) Show that the number of partitions of a positive integer n into parts that are not multiples of three is

equal to the number of partitions of n in which there are at most two repeats. For example, if $n = 6$, then there are 7 partitions of the first kind, namely

$$1{+}1{+}1{+}1{+}1{+}1 = 1{+}1{+}1{+}1{+}2 = 1{+}1{+}2{+}2 = 1{+}1{+}4 = 1{+}5 = 2{+}2{+}2 = 2{+}4$$

There are also 7 partitions of the second kind, namely

$$6 = 1 + 1 + 4 = 1 + 1 + 2 + 2 = 1 + 2 = 3 = 1 + 5 = 2 + 4 = 3 + 3$$